

# CURVATURE PROPERTIES OF 4-DIMENSIONAL RIEMANNIAN MANIFOLDS WITH A CIRCULANT STRUCTURE

IVA DOKUZOVA

**ABSTRACT.** We consider a 4-dimensional Riemannian manifold  $M$  equipped with a circulant structure  $q$ , which is an isometry with respect to the metric  $g$  and  $q^4 = \text{id}$ ,  $q^2 \neq \pm \text{id}$ . For such a manifold  $(M, g, q)$  we obtain some assertions for the sectional curvatures of 2-planes. We construct an example of such a manifold on a Lie group and we find some of its geometric characteristics.

**Mathematics Subject Classification (2010):** 53C15, 53B20, 15B05, 22E60

**Keywords:** Riemannian manifold, Riemannian metric, sectional curvature, circulant matrix, Lie group, Killing metric

## INTRODUCTION

The circulant matrices are well-studied (for example [1], [4]). They have application to Vibration analysis, Graph theory, Linear codes, Geometry (for example [6], [7], [9]).

The study of manifolds with additional structures plays an important role in differential geometry. In such manifolds substantial results are associated with the sectional curvatures of some characteristic 2-planes of the tangent space of the manifolds (for example [2], [3], [10]).

In the present paper we consider some curvature properties of 4-dimensional Riemannian manifolds with a circulant structure  $q$  with  $q^4 = \text{id}$ , which is an isometry with respect to the metric  $g$ . We continue research made in [8] for such manifolds and construct an example of these manifolds.

The paper is organized as follows. In Sect. 1 we give some necessary facts from [8] about a 4-dimensional differentiable manifold  $M$  with a Riemannian metric  $g$ , equipped with a circulant structure  $q$ , which is an isometry with respect to the metric  $g$  and  $q^4 = \text{id}$ ,  $q^2 \neq \pm \text{id}$ . In Sect. 2 we establish that the sectional curvatures of the 2-planes  $\{u, qu\}$  and  $\{u, q^2u\}$  are expressed by the angles  $\angle(u, qu)$  and  $\angle(u, q^2u)$ , respectively. The main results here are Theorem 2.3 and Theorem 2.4. We obtain relations between the sectional curvatures of some characteristic 2-planes in the tangent space on the manifold  $(M, g, q)$ . In Sect. 3 we construct an example of such a manifold on a Lie group and we find some of its geometric characteristics.

## 1. PRELIMINARIES

In this section we recall facts from [8], which are necessary for our future consideration.

Let  $M$  be a 4-dimensional Riemannian manifold with a metric  $g$ . Let  $q$  be an endomorphism in the tangent space  $T_p M$ ,  $p \in M$  on the manifold  $M$  with local coordinates given by the circulant matrix

$$(1) \quad (q_i^j) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$(2) \quad q^4 = \text{id}, \quad q^2 \neq \pm \text{id}.$$

We suppose that  $g$  is positive definite metric and the structure  $q$  of the manifold  $M$  is an isometry with respect to the metric  $g$ , i.e.

$$(3) \quad g(qx, qy) = g(x, y).$$

Anywhere in this work  $x, y, z, u$  will stand for arbitrary elements of the algebra of the smooth vector fields on  $M$  or vectors in the tangent space  $T_p M$ . The Einstein summation convention is used, the range of the summation indices being always  $\{1, 2, 3, 4\}$ .

We denote by  $(M, g, q)$  the manifold  $M$  equipped with the metric  $g$  and the structure  $q$ .

Easily finding that (1) and (3) imply a circulant matrix of components of  $g$ .

A basis of type  $\{x, qx, q^2x, q^3x\}$  of  $T_p M$  is called a  $q$ -basis. In this case we say that *the vector  $x$  induces a  $q$ -basis of  $T_p M$* .

If a vector  $x$  induces a  $q$ -basis, then for the angles  $\angle(x, qx)$ ,  $\angle(x, q^2x)$ ,  $\angle(qx, q^2x)$ ,  $\angle(qx, q^3x)$ ,  $\angle(x, q^3x)$  and  $\angle(q^2x, q^3x)$  we have

$$\angle(x, qx) = \angle(qx, q^2x) = \angle(x, q^3x) = \angle(q^2x, q^3x), \quad \angle(x, q^2x) = \angle(qx, q^3x).$$

In our further research we will use an orthogonal  $q$ -basis. The existence of such bases is proved in [8].

## 2. SOME CURVATURE PROPERTIES

Let  $\nabla$  be the Riemannian connection of the metric  $g$  on  $(M, g, q)$ . The curvature tensor  $R$  of  $\nabla$  is determined by  $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z$ . The tensor of type  $(0, 4)$  associated with  $R$  is defined as follows

$$R(x, y, z, u) = g(R(x, y)z, u).$$

If we denote  $P = q^2$ , then the conditions (2) and (3) imply  $P^2 = \text{id}$ ,  $P \neq \pm \text{id}$ ,  $g(Px, Py) = g(x, y)$ . Thus,  $(M, g, P)$  is a Riemannian manifold with an almost product structure  $P$ . It follows from (1) that  $\text{tr} P = 0$ . For such manifolds is valid Staikova-Gribachev classification ([11]). The class

$W_0$  defined by  $\nabla P = 0$  in this classification is common to all classes. Every manifold in this class satisfies the identity  $R(x, y, Pz, Pu) = R(x, y, z, u)$ . In [8] it is proved analogous identity

$$(4) \quad R(x, y, qz, qu) = R(x, y, z, u),$$

for a manifold  $(M, g, q)$  with the condition  $\nabla q = 0$ . By using (4) and the symmetries of  $R$ , it is easy to find that

$$(5) \quad R(qx, qy, qz, qu) = R(x, y, z, u).$$

Since the latter equality follows from (4), the class of manifolds  $(M, g, q)$  with the condition (5) is more general than the class  $(M, g, q)$  with the condition (4).

If  $\{x, y\}$  is a non-degenerate 2-plane spanned by vectors  $x, y \in T_p M$ , then its sectional curvature is ([12])

$$(6) \quad \mu(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)}.$$

**Theorem 2.1.** *Let  $(M, g, q)$  be a manifold with property (5). If a vector  $x$  induces a  $q$ -basis, then for the sectional curvatures of the basic 2-planes we have*

$$(7) \quad \mu(x, qx) = \mu(qx, q^2x) = \mu(q^2x, q^3x) = \mu(q^3x, x),$$

$$(8) \quad \mu(x, q^2x) = \mu(qx, q^3x).$$

*Proof.* From (5) we have

$$(9) \quad R(x, y, z, u) = R(qx, qy, qz, qu) = R(q^2x, q^2y, q^2z, q^2u).$$

In (9) we substitute

1)  $qx$  for  $y$ ,  $x$  for  $z$ ,  $qx$  for  $u$ , and we get

$$(10) \quad R(x, qx, x, qx) = R(qx, q^2x, qx, q^2x) = R(q^2x, q^3x, q^2x, q^3x),$$

2)  $q^3x$  for  $y$ ,  $x$  for  $z$ ,  $q^3x$  for  $u$ , and we obtain

$$(11) \quad R(x, q^3x, x, q^3x) = R(x, qx, x, qx),$$

3)  $q^2x$  for  $y$ ,  $x$  for  $z$ ,  $q^2x$  for  $u$ , then

$$(12) \quad R(x, q^2x, x, q^2x) = R(qx, q^3x, qx, q^3x).$$

The equality (7) follows from (3), (6), (10) and (11). In a similar way, from (3), (6) and (12) we get (8).  $\square$

Let  $x$  induce a  $q$ -basis  $\{x, qx, q^2x, q^3x\}$ . Due to Theorem 2.1 there are only two different basic sectional curvatures. Therefore, we consider only the sectional curvatures  $\mu(x, qx)$  and  $\mu(x, q^2x)$ . Let us note that if  $y \in \{x, qx\}$  and  $y \neq x$ , then  $qy \notin \{x, qx\}$ . Consequently, we can say that the sectional curvature  $\mu(x, qx)$  depends on  $\varphi = \angle(x, qx)$ . Analogously,  $\mu(x, q^2x)$  depends on  $\theta = \angle(x, q^2x)$ .

We denote  $\mu(x, qx) = \mu_1(\varphi)$  and  $\mu(x, q^2x) = \mu_2(\theta)$ .

**Theorem 2.2.** *Let  $(M, g, q)$  be a manifold with property (5). If vectors  $x$  and  $u$  induce  $q$ -bases and  $\{x, qx, q^2x, q^3x\}$  is orthonormal, then*

$$(13) \quad \begin{aligned} \mu_1(\varphi) - \mu_1\left(\frac{\pi}{2}\right) &= \frac{\cos \varphi}{1 - \cos^2 \varphi} \left( -2R(x, qx, q^2x, x) \right. \\ &\quad + 2(\cos \varphi)R(x, qx, qx, q^2x) \\ &\quad - (\cos \varphi)R(qx, q^2x, q^3x, x) \\ &\quad \left. - 2R(qx, q^2x, q^2x, x) \right), \end{aligned}$$

$$(14) \quad \begin{aligned} \mu_2(\theta) - \mu_2\left(\frac{\pi}{2}\right) &= \frac{2 \cos \theta}{1 - \cos^2 \theta} \left( -2R(x, qx, qx, q^2x) \right. \\ &\quad \left. + (\cos \theta)R(qx, q^2x, q^3x, x) \right), \end{aligned}$$

where  $\varphi = \angle(u, qu)$ ,  $\theta = \angle(u, q^2u)$ .

*Proof.* In (9) we substitute

1)  $qx$  for  $y$ ,  $q^2x$  for  $z$  and  $x$  for  $u$ , then

$$(15) \quad R(x, qx, q^2x, x) = R(q^2x, q^3x, x, q^2x) = R(q^3x, x, qx, q^3x),$$

2)  $qx$  for  $y$ ,  $qx$  for  $z$  and  $q^2x$  for  $u$ , and we have

$$(16) \quad R(x, qx, qx, q^2x) = R(q^2x, q^3x, q^3x, x) = R(q^3x, x, x, qx),$$

3)  $qx$  for  $y$ ,  $q^2x$  for  $z$  and  $q^3x$  for  $u$ , then

$$(17) \quad R(qx, q^2x, q^3x, x) = R(x, qx, q^2x, q^3x),$$

4)  $qx$  for  $y$ ,  $qx$  for  $z$  and  $q^3x$  for  $u$ , and we get

$$(18) \quad R(qx, q^2x, q^2x, x) = R(x, qx, qx, q^3x) = R(q^3x, x, x, q^2x),$$

5)  $q^2x$  for  $y$ ,  $qx$  for  $z$  and  $q^3x$  for  $u$ , and we find

$$(19) \quad R(x, q^2x, qx, q^3x) = 0.$$

Let  $u = \alpha x + \beta qx + \gamma q^2x + \delta q^3x$ , where  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ . From (1) we get  $qu = \delta x + \alpha qx + \beta q^2x + \gamma q^3x$ ,  $q^2u = \gamma x + \delta qx + \alpha q^2x + \beta q^3x$  and  $q^3u = \beta x + \gamma qx + \delta q^2x + \alpha q^3x$ . Then, by using the linear properties of the

curvature tensor  $R$  and having in mind (9), (11), (12), (15) – (19), we obtain

$$\begin{aligned}
R(u, qu, u, qu) &= \left( (\alpha^2 - \beta\delta)^2 + (\delta^2 - \alpha\gamma)^2 + (\beta^2 - \alpha\gamma)^2 + (\gamma^2 - \beta\delta)^2 \right) R_1 \\
&\quad + 2 \left( (\alpha\beta - \gamma\delta)(\gamma^2 - \alpha^2) + (\beta\gamma - \delta\alpha)(\delta^2 - \beta^2) \right) R_2 \\
&\quad + \left( (\alpha\beta - \gamma\delta)^2 + (\beta\gamma - \delta\alpha)^2 \right) R_3 \\
&\quad + 2(\alpha^2 + \gamma^2 - 2\beta\delta)(\delta^2 + \beta^2 - 2\alpha\gamma) R_4 \\
&\quad + 2 \left( (\alpha^2 - \beta\delta)(\gamma^2 - \beta\delta) + (\beta^2 - \alpha\gamma)(\delta^2 - \alpha\gamma) \right) R_5 \\
&\quad + 2 \left( (\alpha\beta - \gamma\delta)(\delta^2 - \beta^2) + (\beta\gamma - \delta\alpha)(\alpha^2 - \gamma^2) \right) R_6, \\
R(u, q^2u, u, q^2u) &= 2 \left( (\alpha\delta - \beta\gamma)^2 + (\alpha\beta - \gamma\delta)^2 \right) R_1 \\
&\quad + 4 \left( (\alpha\delta - \beta\gamma)(\gamma^2 - \alpha^2) + (\alpha\beta - \delta\gamma)(\delta^2 - \beta^2) \right) R_2 \\
&\quad + \left( (\alpha^2 - \gamma^2)^2 + (\beta^2 - \delta^2)^2 \right) R_3 \\
&\quad - 2 \left( (\alpha\beta - \gamma\delta)^2 + (\beta\gamma - \alpha\delta)^2 \right) R_5 \\
&\quad + 4 \left( (\beta^2 - \delta^2)(\alpha\delta - \beta\gamma) + (\alpha\beta - \gamma\delta)(\gamma^2 - \alpha^2) \right) R_6,
\end{aligned}$$

where

$$\begin{aligned}
(20) \quad R_1 &= R(x, qx, x, qx), \quad R_2 = R(x, qx, q^2x, x), \\
R_3 &= R(x, q^2x, x, q^2x), \quad R_4 = R(x, qx, qx, q^2x), \\
R_5 &= R(qx, q^2x, q^3x, x), \quad R_6 = R(qx, q^2x, q^2x, x).
\end{aligned}$$

Then

$$\begin{aligned}
(21) \quad R(u, qu, u, qu) + \frac{1}{2}R(u, q^2u, u, q^2u) &= K_1R_1 + K_2R_2 + K_3R_3 \\
&\quad + K_4R_4 + K_5R_5 + K_6R_6,
\end{aligned}$$

where

$$\begin{aligned}
(22) \quad K_1 &= (\alpha^2 - \beta\delta)^2 + (\delta^2 - \alpha\gamma)^2 + (\beta^2 - \alpha\gamma)^2 + (\gamma^2 - \beta\delta)^2 \\
&\quad + (\alpha\delta - \beta\gamma)^2 + (\alpha\beta - \gamma\delta)^2, \\
K_2 &= 2 \left( (\alpha\beta - \gamma\delta)(\gamma^2 - \alpha^2) + (\beta\gamma - \delta\alpha)(\delta^2 - \beta^2) \right. \\
&\quad \left. + (\alpha\delta - \beta\gamma)(\gamma^2 - \alpha^2) + (\alpha\beta - \delta\gamma)(\delta^2 - \beta^2) \right), \\
K_3 &= (\alpha\beta - \gamma\delta)^2 + (\beta\gamma - \delta\alpha)^2 \\
&\quad + \frac{1}{2} \left( (\alpha^2 - \gamma^2)^2 + (\beta^2 - \delta^2)^2 \right),
\end{aligned}$$

$$\begin{aligned}
K_4 &= 2(\alpha^2 + \gamma^2 - 2\beta\delta)(\delta^2 + \beta^2 - 2\alpha\gamma), \\
K_5 &= 2\left((\alpha^2 - \beta\delta)(\gamma^2 - \beta\delta) + (\beta^2 - \alpha\gamma)(\delta^2 - \alpha\gamma)\right) \\
&\quad - (\alpha\beta - \gamma\delta)^2 - (\beta\gamma - \alpha\delta)^2.
\end{aligned}$$

Since the  $q$ -basis  $\{x, qx, q^2x, q^3x\}$  is orthonormal, we have

$$g(u, u) = g(qu, qu) = \alpha^2 + \beta^2 + \gamma^2 + \delta^2,$$

$$g(u, qu) = \alpha\delta + \alpha\beta + \beta\gamma + \delta\gamma, \quad g(u, q^2u) = 2(\alpha\gamma + \delta\beta).$$

Due to (3) and (6) we get

$$\mu(u, qu) = \frac{R(u, qu, u, qu)}{g^2(u, u) - g^2(u, qu)}, \quad \mu(u, q^2u) = \frac{R(u, q^2u, u, q^2u)}{g^2(u, u) - g^2(u, q^2u)}.$$

We suppose that  $g(u, u) = 1$  and we obtain

$$(23) \quad \mu(u, qu) = \frac{R(u, qu, u, qu)}{1 - \cos^2 \varphi}, \quad \mu(u, q^2u) = \frac{R(u, q^2u, u, q^2u)}{1 - \cos^2 \theta},$$

$$(24) \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1, \quad \alpha\delta + \alpha\beta + \beta\gamma + \delta\gamma = \cos \varphi, \quad 2(\alpha\gamma + \delta\beta) = \cos \theta.$$

From (24) we express  $\alpha, \beta, \gamma, \delta$  by  $\cos \varphi$  and  $\cos \theta$ . Then, taking into account (22), we get

$$\begin{aligned}
K_1 &= 1 - \cos^2 \varphi, \quad K_2 = -2 \cos \varphi (1 - \cos \theta), \quad K_3 = \frac{1}{2}(1 - \cos^2 \theta), \\
K_4 &= 2(-\cos \theta + \cos^2 \varphi), \quad K_5 = \cos^2 \theta - \cos^2 \varphi.
\end{aligned}$$

Thus, (21) and (23) imply

$$\begin{aligned}
(25) \quad &(1 - \cos^2 \varphi)\mu(u, qu) + \frac{1}{2}(1 - \cos^2 \theta)\mu(u, q^2u) = \\
&(1 - \cos^2 \varphi)R_1 - 2 \cos \varphi (1 - \cos \theta)R_2 \\
&+ \frac{1}{2}(1 - \cos^2 \theta)R_3 + 2(-\cos \theta + \cos^2 \varphi)R_4 \\
&+ (\cos^2 \theta - \cos^2 \varphi)R_5 - 2 \cos \varphi (1 - \cos \theta)R_6.
\end{aligned}$$

In (25) first we substitute  $\varphi = \frac{\pi}{2}$  and then  $\theta = \frac{\pi}{2}$ . Thus we obtain (14) and (13).  $\square$

**Theorem 2.3.** *Let  $(M, g, q)$  be a manifold with property (5). If a vector  $u$  induces a  $q$ -basis, then the following equality is valid*

$$\begin{aligned}
(26) \quad \mu_1(\varphi) &= \frac{1}{1 - \cos^2 \varphi} \left( (1 - 4 \cos^2 \varphi) \mu_1\left(\frac{\pi}{2}\right) \right. \\
&\quad + \frac{3}{4}(\cos \varphi + 2 \cos^2 \varphi) \mu_1\left(\frac{\pi}{3}\right) \\
&\quad \left. + \frac{3}{4}(2 \cos^2 \varphi - \cos \varphi) \mu_1\left(\frac{2\pi}{3}\right) \right),
\end{aligned}$$

where  $\varphi = \angle(u, qu)$ .

*Proof.* In (13) first we substitute  $\varphi = \frac{\pi}{3}$  and then  $\varphi = \frac{2\pi}{3}$ . Due to (20) and having in mind that  $\{x, qx, q^2x, q^3x\}$  is an orthonormal  $q$ -basis, we get

$$\begin{aligned} 3\left(\mu_1\left(\frac{\pi}{3}\right) - \mu_1\left(\frac{\pi}{2}\right)\right) &= 4\left(-R_2 + \frac{1}{2}R_4 - \frac{1}{4}R_5 - R_6\right), \\ 3\left(\mu_1\left(\frac{2\pi}{3}\right) - \mu_1\left(\frac{\pi}{2}\right)\right) &= 4\left(R_2 + \frac{1}{2}R_4 - \frac{1}{4}R_5 + R_6\right). \end{aligned}$$

From the latter equalities we find the tensors  $R_4 - \frac{1}{2}R_5$  and  $R_2 + R_6$ . Then (13) implies (26).  $\square$

**Theorem 2.4.** *Let  $(M, g, q)$  be a manifold with property (5). If a vector  $u$  induces a  $q$ -basis, then the following equality is valid*

$$\begin{aligned} \mu_2(\theta) &= \frac{1}{1 - \cos^2 \theta} \left( (1 - 4 \cos^2 \theta) \mu_2\left(\frac{\pi}{2}\right) \right. \\ (27) \quad &+ \frac{3}{4}(\cos \theta + 2 \cos^2 \theta) \mu_2\left(\frac{\pi}{3}\right) \\ &\left. + \frac{3}{4}(2 \cos^2 \theta - \cos \theta) \mu_2\left(\frac{2\pi}{3}\right) \right), \end{aligned}$$

where  $\theta = \angle(u, q^2u)$ .

*Proof.* In (14) first we substitute  $\theta = \frac{\pi}{3}$  and then  $\theta = \frac{2\pi}{3}$ . Thus we get

$$\begin{aligned} 3\left(\mu_2\left(\frac{\pi}{3}\right) - \mu_2\left(\frac{\pi}{2}\right)\right) &= 2R_5 - 8R_4, \\ 3\left(\mu_2\left(\frac{2\pi}{3}\right) - \mu_2\left(\frac{\pi}{2}\right)\right) &= 2R_5 + 8R_4. \end{aligned}$$

Taking into account the last system and (14), we obtain (27).  $\square$

**Theorem 2.5.** *Let  $(M, g, q)$  be a manifold with property (4). If a vector  $u$  induces a  $q$ -basis, then the following equalities are valid*

$$(28) \quad \mu_2(\theta) = 0, \quad \mu_1(\varphi) = \frac{(1 - \cos \theta)^2}{1 - \cos^2 \varphi} \mu_1\left(\frac{\pi}{2}\right),$$

where  $\theta = \angle(u, q^2u)$  and  $\varphi = \angle(u, qu)$ .

*Proof.* From (4) we have

$$(29) \quad R(x, y, qz, qu) = R(x, y, q^2z, q^2u).$$

In (4), (29) we substitute

1)  $qx$  for  $y$ ,  $x$  for  $z$ ,  $qx$  for  $u$ , and we get

$$(30) \quad R(x, qx, x, qx) = R(x, qx, qx, q^2x) = R(x, qx, q^2x, q^3x),$$

2)  $q^2x$  for  $y$ ,  $x$  for  $z$ ,  $q^2x$  for  $u$ , and we obtain

$$(31) \quad R(x, q^2x, x, q^2x) = R(x, q^2x, qx, q^3x) = R(x, q^2x, q^2x, x),$$

3)  $qx$  for  $y$ ,  $q^2x$  for  $z$ ,  $x$  for  $u$ , then

$$(32) \quad R(x, qx, q^2x, x) = R(x, qx, q^3x, qx) = R(x, qx, x, q^2x).$$

Comparing the identities (17), (18), (30), (31), (32) and having in mind (20), we obtain  $R_1 = R_4 = R_5$  and  $R_2 = R_3 = R_6 = 0$ . Then (25) implies (28).  $\square$

We note that Proposition 4.2 from [8] is a particular case of Theorem 2.5.

### 3. A LIE GROUP AS A 4-DIMENSIONAL RIEMANNIAN MANIFOLD WITH A CIRCULANT STRUCTURE

Let  $G$  be a 4-dimensional real connected Lie group and  $\mathfrak{g}$  be its Lie algebra with a basis  $\{x_1, x_2, x_3, x_4\}$ . We introduce a structure  $q$  and left invariant metric  $g$  as follows

$$(33) \quad qx_1 = x_2, \quad qx_2 = x_3, \quad qx_3 = x_4, \quad qx_4 = x_1,$$

$$(34) \quad g(x_i, x_j) = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$$

Obviously, (1) and (3) are valid. Therefore  $(G, g, q)$  is a Riemannian manifold with (1) and (3).

For the manifold  $(G, g, q)$  we suppose that  $g$  is a Killing metric, i.e.

$$(35) \quad g([x_i, x_j], x_k) + g([x_i, x_k], x_j) = 0.$$

According to (35) and the Jacobi identity for the commutators  $[x_i, x_j]$  we obtain

$$(36) \quad \begin{aligned} [x_1, x_2] &= \lambda_1 x_3 + \lambda_2 x_4, & [x_1, x_3] &= -\lambda_1 x_2 + \lambda_4 x_4, \\ [x_2, x_3] &= \lambda_1 x_1 + \lambda_3 x_4, & [x_1, x_4] &= -\lambda_2 x_2 - \lambda_4 x_3, \\ [x_2, x_4] &= \lambda_2 x_1 - \lambda_3 x_3, & [x_3, x_4] &= \lambda_4 x_1 + \lambda_3 x_2, \end{aligned}$$

where  $\lambda_i \in \mathbb{R}$ .

Vice versa, if (36) are valid for a Riemannian manifold  $(G, g, q)$ , where the structure  $q$  and the metric  $g$  on the Lie group  $G$  are determined by (1) and (3), then the Jacobi identity for commutators  $[x_i, x_j]$  is satisfied and the metric  $g$  is Killing.

Therefore, we establish the truthfulness of the following

**Theorem 3.1.** *Let  $(G, g, q)$  be a 4-dimensional Riemannian manifold, where  $G$  is the connected Lie group with an associated Lie algebra  $\mathfrak{g}$ , determined by a global basis  $\{x_i\}$  of left invariant vector fields, and  $q$  and  $g$  are the structure and the metric, determined by (33) and (34). Then  $(G, g, q)$  is a Riemannian manifold with a circulant structure  $q$  and a Killing metric  $g$ , which satisfy (1) and (3) if and only if  $G$  belongs to a Lie group, determined by (36).*

Further,  $(G, g, q)$  will stand for the Riemannian manifold determined by the conditions of Theorem 3.1.

Since  $g$  is a Killing metric, then the components of  $R$  are ([5])



$$(37) \quad R_{ijkh} = -\frac{1}{4}g\left([x_i, x_j], [x_k, x_h]\right).$$

According to (34), (36) and (37) we calculate the following components of the curvature tensor  $R$ :

$$(38) \quad \begin{aligned} R_{1212} &= -\frac{1}{4}(\lambda_1^2 + \lambda_2^2), & R_{1414} &= -\frac{1}{4}(\lambda_2^2 + \lambda_4^2), \\ R_{2323} &= -\frac{1}{4}(\lambda_1^2 + \lambda_3^2), & R_{3434} &= -\frac{1}{4}(\lambda_3^2 + \lambda_4^2), \\ R_{1313} &= -\frac{1}{4}(\lambda_1^2 + \lambda_4^2), & R_{2424} &= -\frac{1}{4}(\lambda_2^2 + \lambda_3^2), \\ R_{1213} &= R_{2434} = -\frac{1}{4}\lambda_2\lambda_4, & R_{2324} &= R_{1314} = -\frac{1}{4}\lambda_1\lambda_2, \\ R_{1424} &= R_{1323} = -\frac{1}{4}\lambda_3\lambda_4, & R_{3134} &= R_{2124} = -\frac{1}{4}\lambda_1\lambda_3, \\ R_{1214} &= R_{3234} = \frac{1}{4}\lambda_1\lambda_4, & R_{1434} &= R_{2123} = \frac{1}{4}\lambda_2\lambda_3. \end{aligned}$$

The rest of nonzero components are obtained from the properties

$$R_{ijks} = R_{ksij}, \quad R_{ijks} = -R_{jiks} = -R_{ijsk}.$$

**Proposition 3.2.** *Let  $(G, g, q)$  be a manifold determined by the conditions of Theorem 3.1. Then  $(G, g, q)$  satisfies the identity (5) if and only if*

$$(39) \quad \lambda_1 = \varepsilon\lambda_2 = \varepsilon\lambda_3 = \lambda_4, \quad \varepsilon = \pm 1.$$

*Proof.* According to (33) we obtain that (5) is equivalent to

$$\begin{aligned} R_{1212} &= R_{3434} = R_{2323} = R_{1414}, & R_{1313} &= R_{2424}, \\ R_{1213} &= R_{2324} = R_{1424} = R_{3134}, & R_{1214} &= R_{1434} = R_{2123} = R_{3234}, \\ R_{1224} &= R_{3123} = R_{3114} = R_{4234}, & R_{1324} &= 0. \end{aligned}$$

Then (38) implies

$$\lambda_2 = \lambda_3, \quad \lambda_1 = \lambda_4, \quad \lambda_1^2 + \lambda_4^2 = \lambda_2^2 + \lambda_3^2, \quad \lambda_1\lambda_4 = \lambda_2\lambda_3.$$

So we obtain (39). □

From (38) and (39) we calculate

$$(40) \quad \begin{aligned} R_{1212} &= R_{1414} = R_{2323} = R_{3434} = R_{1313} = R_{2424} = -\frac{1}{2}\lambda_1^2, \\ R_{1213} &= R_{2434} = R_{2324} = R_{1314} = R_{1424} = \\ &= R_{3431} = R_{1323} = R_{2124} = -\frac{1}{4}\varepsilon\lambda_1^2, \\ R_{1214} &= R_{3234} = R_{1434} = R_{2123} = \frac{1}{4}\lambda_1^2. \end{aligned}$$

Having in mind (40) and the formulas

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j),$$

we get the components of the Ricci tensor  $\rho$  and the value of the scalar curvature  $\tau$  as follows:

$$(41) \quad \begin{aligned} \rho_{11} &= \rho_{22} = \rho_{33} = \rho_{44} = \frac{3}{2} \lambda_1^2, \\ \rho_{12} &= \rho_{14} = \rho_{23} = \rho_{34} = \frac{1}{2} \varepsilon \lambda_1^2, \\ \rho_{13} &= \rho_{24} = -\frac{1}{2} \lambda_1^2, \end{aligned}$$

$$(42) \quad \tau = 6 \lambda_1^2.$$

By using (6) for the sectional curvatures of the basic 2-planes we find

$$(43) \quad \mu(x_1, x_2) = \mu(x_1, x_4) = \mu(x_2, x_3) = \mu(x_2, x_4) = \mu(x_1, x_3) = -\frac{1}{2} \lambda_1^2.$$

Therefore, we arrive at the following

**Proposition 3.3.** *Let  $(G, g, q)$  be a manifold determined by the conditions of Theorem 3.1. If  $(G, g, q)$  satisfies the identity (5), then*

- (i) *The nonzero components of the curvature tensor  $R$  and the Ricci tensor  $\rho$  are (40) and (41);*
- (ii) *The scalar curvature  $\tau$  is (42);*
- (iii)  *$(G, g, q)$  is of constant sectional curvatures (43), i.e.  $(G, g, q)$  is conformally flat manifold.*

According to (33) we obtain that (4) is equivalent to the equalities

$$\begin{aligned} R_{1212} &= R_{1414} = R_{2323} = R_{3434} = R_{1223} = R_{1241} = \\ &R_{4134} = R_{1234} = R_{2334} = R_{2341}, \\ R_{1313} &= R_{2424} = R_{1324} = R_{1213} = R_{1224} = R_{1431} = \\ &R_{2441} = R_{2423} = R_{2331} = R_{1334} = R_{2434} = 0. \end{aligned}$$

Then, by using (38) and (42) we have the following

**Proposition 3.4.** *Let  $(G, g, q)$  be a manifold determined by the conditions of Theorem 3.1. Then the following propositions are equivalent:*

- (i)  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ , i.e.  $G$  is abelian;
- (ii)  $(G, g, q)$  satisfies the identity (4);
- (iii)  $\tau = 0$ , i.e.  $(G, g, q)$  is a scalar flat manifold with respect to  $\nabla$ .

#### ACKNOWLEDGMENTS

This work was partially supported by project NI15-FMI-004 of the Scientific Research Fund, Paisii Hilendarski University of Plovdiv, Bulgaria.

## REFERENCES

- [1] DAVIS, P. J.: Circulant matrices, New York, John Wiley and Sons, 250 (1979)
- [2] GADEA, P. M., MONTESINOS, A.: Spaces of constant para-holomorphic sectional curvature, Pacific J. Math., **136**, No 1, 85–101 (1989)
- [3] GRAY, A., VANHECKE, L.: Almost Hermitian manifolds with constant holomorphic sectional curvature, Appl. Math., **104**, 170–179 (1979)
- [4] GRAY, R. M.: Toeplitz and circulant matrices: A review, Found. Trends Commun. Inf. Theory, **2**, No 3, 155–239 (2006)
- [5] MEKEROV, D.: Lie groups as 4-dimensional Riemannian or pseudo-Riemannian almost product manifolds with nonintegrable structure, J. Geom., **90**, 165–174 (2008)
- [6] MUZYCHUK, M.: A solution of the isomorphism problem for circulant graphs, In: Proc. London Math. Soc., **88**, No 3, 1–41 (2004)
- [7] OLSON, B., SHAW, S., SHI, C., PIERRE C., PARKER, R. G.: Circulant matrices and their application to vibration analysis, Appl. Mech. Rev., **66**, No 4, 1–41 (2014)
- [8] RAZPOPOV, D.: Four-dimensional Riemannian manifolds with two circulant structures, In: Proc. of 44-th Spring Conf. of UBM, SOK "Kamchia", Bulgaria, 179–185 (2015)
- [9] ROTH, R. M., LEMPEL, A.: Application of circulant matrices to the construction and decoding of linear codes, IEEE Trans. Inf. Theory, **36**, No 5, 1157–1163 (1990)
- [10] STAIKOVA, M., GRIBACHEV, K., MEKEROV, D.: Riemannian  $P$ -manifolds of constant sectional curvatures, Serdica Math. J., **17**, 212–219 (1991)
- [11] STAIKOVA, M., GRIBACHEV, K.: Canonical connections and their conformal invariants on Riemannian  $P$ -manifolds, Serdica Math. J., **18**, 150–161 (1992)
- [12] YANO, K.: Differential geometry on complex and almost complex spaces. Pure and Applied Math. **49**, New York, Pergamont Press Book, 326 (1965)